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Generalized Nonlinear Implicit Quasi-Variational Inclusions with Fuzzy Mappings

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Abstract—In this paper, we introduce and study a new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings. An existence theorem of solutions is proved without compactness assumptions. A new iterative algorithm is suggested and analysed. The convergence of the iterative sequence generated by the new algorithm is also given. As special cases, some known results are also discussed. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Generalized nonlinear implicit quasivariational inclusion with fuzzy mappings, Maximal monotone mapping, Iterative algorithm, Hilbert space.

1. INTRODUCTION

In 1994, Hassouni and Moudafi [1] introduced a class of variational inclusions which included many variational inequalities and quasivariational inequalities as special cases. Adly [2], Huang [3], Kazmi [4], and Ding [5,6] extended the results in [1] to general variational inclusions and to generalized quasivariational inclusions.

In 1989, Chang and Zhu [7] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then, several classes of variational inequalities with fuzzy mappings were considered by Chang and Huang [8], Noor [9], and Huang [10].

In 1992 and 1997, by studying an elastoplasticity problem, Panagiotopoulos and Stavroulakis [11] and Noor and Al-Said [12] considered a new class of generalized nonlinear variational inequality problems, which is a variant form and generalization of the problem proposed by Verma [13,14]. In 1999, Ding [15] considered a class of generalized implicit quasivariational inclusions with fuzzy set-valued mappings which includes many known variational and quasivariational inclusions as special cases.

In this paper, we shall introduce and study a new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings, which includes many new and known classes of generalized variational inequalities, generalized quasivariational inequalities, and generalized quasivariational inclusions as special cases. An existence theorem of solutions is proved without compactness assumptions. A new iterative algorithm for finding approximate solutions is proposed and analysed. The convergence of the iterative sequence generated by the new algorithm is also given. As special cases, some known results in the field are also discussed.

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2. PRELIMINARIES

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(H)$ be a collection of all fuzzy sets over H. A mapping $F : H \to \mathcal{F}(H)$ is said to be a fuzzy mapping. For each $x \in H$, $F(x)$ (denote it by F_x in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : H \to \mathcal{F}(H)$ is said to be closed if for each $x \in H$, the function $y \mapsto F_x(y)$ is upper semicontinuous, i.e., for any given net $\{y_\alpha\} \subset H$ satisfying $y_\alpha \to y_0 \in H$, $\limsup_\alpha F_x(y_\alpha) \leq F_x(y_0)$. For $B \in \mathcal{F}(H)$ and $\lambda \in [0,1]$, the set $(B)_\lambda = \{x \in H : B(x) \geq \lambda\}$ is called a λ-cut set of B.

A closed fuzzy mapping $E : H \to \mathcal{F}(H)$ is said to satisfy Condition (A): if there exists a function $a : H \to [0,1]$ such that for each $x \in H$, $(E_x)_{a(x)}$ is a nonempty bounded subset of H. It is clear that if E is a closed fuzzy mapping satisfying Condition (A), then for each $x \in H$, the set $(E_x)_{a(x)} \in \mathrm{CB}(H)$, where $\mathrm{CB}(H)$ denotes the family of all nonempty bounded closed subsets of H. In fact, let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (E_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in H$. Then $(E_x)(y_\alpha) \geq a(x)$ for each $\alpha \in \Gamma$. Since E is closed, we have

$$E_x(y_0) \geq \limsup_{\alpha\in\Gamma} E_x(y_\alpha) \geq a(x).$$

This implies $y_0 \in (E_x)_{a(x)}$ and so $(E_x)_{a(x)} \in \mathrm{CB}(H)$.

Let $E, F, G, D : H \to \mathcal{F}(H)$ be four closed fuzzy mappings satisfying Condition (A). Then there exist four functions $a, b, c, d : H \to [0,1]$ such that for each $x \in H$, we have $(E_x)_{a(x)}, (F_x)_{b(x)}, (G_x)_{c(x)}, (D_x)_{d(x)} \in \mathrm{CB}(H)$. Therefore, we can define four set-valued mappings $\tilde{E}, \tilde{F}, \tilde{G}, \tilde{D} : H \to \mathrm{CB}(H)$ by

$$\tilde{E}(x) = (E_x)_{a(x)}, \qquad \tilde{F}(x) = (F_x)_{b(x)}, \qquad \tilde{G}(x) = (G_x)_{c(x)}, \qquad \tilde{D}(x) = (D_x)_{d(x)},$$

for each $x \in H$. In the sequel, $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, and $\tilde{D}$ are called the set-valued mappings induced by the fuzzy mappings E, F, G, and D, respectively.

Let $N : H \times H \to H$ and $g : H \to H$ be single-valued mappings and let $E, F, G, D : H \to \mathcal{F}(H)$ be fuzzy mappings. Let $a, b, c, d : H \to [0,1]$ be given functions. Let $M : H \times H \to 2^H$ be a set-valued mapping such that for each given $y \in H$, $M(\cdot, y) : H \to 2^H$ is a maximal monotone mapping with $g(H) \cap \mathrm{dom}(M(\cdot, y)) \neq \emptyset$. Throughout this paper, unless specially stated, we will consider the following generalized nonlinear implicit quasivariational inclusion problem with fuzzy mappings:

$$\begin{aligned}&\text{find } x, u, v, z, w \in H \text{ such that } E_x(u) \geq a(x),\ F_x(v) \geq b(x),\\ &G_x(z) \geq c(x),\ D_x(w) \geq d(x), \text{ and } 0 \in M(g(x), z) + g(w) - N(u, v).\end{aligned} \tag{2.1}$$

Special Cases

(I) If I is the identity mapping on H, D is the fuzzy mapping defined by $x \mapsto \chi_{\{x\}}$, where $\chi_{\{x\}}$ is the characteristic function of $\{x\}$ and $d(x) = 1$ for all $x \in H$, then problem (2.1) reduces to the following problem:

$$\begin{aligned}&\text{find } x, u, v, z \in H \text{ such that } E_x(u) \geq a(x),\ F_x(v) \geq b(x),\ G_x(z) \geq\\ &c(x), \text{ and } 0 \in M(g(x), z) + g(x) - N(u, v).\end{aligned} \tag{2.2}$$

Problem (2.2) was considered by Ding [15], which was called the generalized implicit quasivariational inclusion with fuzzy mappings.

(II) If $E, F, G, D : H \to \mathrm{CB}(H)$ are classical set-valued mappings, we can define the fuzzy mappings $E, F, G, D : H \to \mathcal{F}(H)$ by

$$x \mapsto \chi_{E(x)}, \qquad x \mapsto \chi_{F(x)}, \qquad x \mapsto \chi_{G(x)}, \qquad x \mapsto \chi_{D(x)},$$

where $\chi_{E(x)}$, $\chi_{F(x)}$, $\chi_{G(x)}$, and $\chi_{D(x)}$ are the characteristic functions of $E(x)$, $F(x)$, $G(x)$, and $D(x)$, respectively. Taking $a(x) = b(x) = c(x) = d(x) = 1$ for all $x \in H$, then problem (2.1) is equivalent to the following problem:

$$\text{find } x \in H,\ u \in E(x),\ v \in F(x),\ z \in G(x), \text{ and } w \in D(x) \text{ such that } 0 \in M(g(x), z) + g(w) - N(u, v). \tag{2.3}$$

Problem (2.3) is a new class of generalized nonlinear implicit quasivariational inclusions which includes many generalized variational and generalized quasivariational inequality problems, generalized variational and generalized quasivariational inclusions studied by Adly [2] and many authors.

(III) Let $\phi : H \times H \to \mathrm{R} \cup \{\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper convex lower semicontinuous functional satisfying $g(H) \cap \operatorname{dom}(\partial\phi(\cdot, y)) \neq \emptyset$, where $\partial\phi(\cdot, y))$ is the subdifferential of $\phi(\cdot, y)$. By [16], $\partial\phi(\cdot, y) : H \to 2^H$ is a maximal monotone mapping. Let $M(x, y) = \partial\phi(x, y)$, $\forall\, x, y \in H$. For given $z \in H$, by the definition of the subdifferential of $\phi(\cdot, z)$, it is easy to see that problem (2.1) reduces to the following problem:

$$\text{find } x, u, v, z, w \in H \text{ such that } E_x(u) \geq a(x),\ F_x(v) \geq b(x),\ G_x(z) \geq c(x),\ D_x(w) \geq d(x), \text{ and } \langle g(w) - N(u, v), y - g(x)\rangle \geq \phi(g(x), z) - \phi(y, z),\ \forall\, y \in H. \tag{2.4}$$

Problem (2.4) is a variant form or generalization of the problems considered in [1,3–6,9,10].

(IV) If $K : H \to 2^H$ is a set-valued mapping such that each $K(x)$ is a closed convex subset of H (or $K(x) = m(x) + K$, where $m : H \to H$ and K is a closed convex subset of H) and for each fixed $z \in H$, $\phi(\cdot, z) = I_{K(z)}(\cdot)$ is the indicator function of $K(z)$,

$$I_{K(z)}(x) = \begin{cases} 0, & \text{if } x \in K(z), \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.4) reduces to the generalized strongly nonlinear implicit quasivariational inequality problem with fuzzy mappings:

$$\text{find } x, u, v, z, w \in H \text{ such that } E_x(u) \geq a(x),\ F_x(v) \geq b(x),\ G_x(z) \geq c(x),\ D_x(w) \geq d(x), \text{ and } g(x) \in K(z),\ \langle g(w) - N(u, v), y - g(x)\rangle \geq 0,\ \forall\, y \in K(z). \tag{2.5}$$

(V) If $N(u, v) = g(v) - u$, $\forall\, u, v \in H$, $G(x)$ and $D(x)$ are both the characteristic function $\chi_{\{x\}}$ of $\{x\}$ and $c(x) = d(x) = 1$ for each $x \in H$, then problem (2.5) reduces to the following problem:

$$\text{find } x, u, v \in H \text{ such that } E_x(u) \geq a(x),\ F_x(v) \geq b(x), \text{ and } g(x) \in K(x),\ \langle g(x) - (g(v) - u), y - g(x)\rangle \geq 0,\ \forall\, y \in K(x). \tag{2.6}$$

Problem (2.6) is called the generalized strongly nonlinear quasivariational inequality problem with fuzzy mappings, which includes the generalized nonlinear variational inequality problem considered by Noor and Al-Said [12] and Verma [13,14] as very special cases.

DEFINITION 2.1. *Let H be a Hilbert space and let $M : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^M : H \to H$ defined by*

$$J_\rho^M(x) = (I + \rho M)^{-1}(x), \qquad \forall\, x \in H,$$

is said to be the resolvent operator of M, where I is the identity mapping on H.

LEMMA 2.1. *(See [16].) Let $M : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^M : H \to H$ of M is nonexpansive, i.e.,*

$$\left\| J_\rho^M(x) - J_\rho^M(y) \right\| \leq \|x - y\|, \qquad \forall\, x, y \in H.$$

DEFINITION 2.2. *A mapping $g : H \to H$ is said to be*

(i) *δ-strongly monotone if there exists a constant $\delta > 0$ such that*

$$\langle g(x) - g(y), x - y\rangle \geq \delta \|x - y\|^2, \qquad \forall\, x, y \in H,$$

(ii) *σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that*

$$\|g(x) - g(y)\| \leq \sigma \|x - y\|, \qquad \forall\, x, y \in H.$$

DEFINITION 2.3. *Let $E : H \to 2^H$ and $N : H \times H \to H$ be mappings.*

(i) *E is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that*

$$\langle u_1 - u_2, x_1 - x_2\rangle \geq \alpha \|x_1 - x_2\|^2, \qquad \forall\, x_1, x_2 \in H, \quad u_1 \in E(x_1), \quad u_2 \in E(x_2).$$

(ii) *$N(\cdot,\cdot)$ is said to be α-relaxed Lipschitz with respect to E in the first argument if there exists a constant $\alpha > 0$ such that*

$$\langle N(u_1,\cdot) - N(u_2,\cdot), x_1 - x_2\rangle \leq -\alpha \|x_1 - x_2\|^2, \qquad \forall\, x_1, x_2 \in H, \quad u_1 \in E(x_1), \quad u_2 \in E(x_2).$$

(iii) *$N(\cdot,\cdot)$ is said to be β-Lipschitz continuous in the first argument if there exists a constant $\beta > 0$ such that*

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leq \beta \|u_1 - u_2\|, \qquad \forall\, u_1, u_2 \in H.$$

In a similar way, we can define the ξ-Lipschitz continuity of $N(\cdot,\cdot)$ in the second argument.

DEFINITION 2.4. *A set-valued mapping $E : H \to \mathrm{CB}(H)$ is said to be ϵ-Lipschitz continuous if there exists a constant $\epsilon > 0$ such that*

$$\tilde{H}(E(x), E(y)) \leq \epsilon \|x - y\|, \qquad \forall\, x, y \in H,$$

where $\tilde{H}(\cdot,\cdot)$ is the Hausdorff metric on $\mathrm{CB}(H)$.

3. EXISTENCE AND ITERATIVE ALGORITHM OF SOLUTIONS

We first transfer problem (2.1) into a fixed-point problem.

THEOREM 3.1. *(x, u, v, z, w) is a solution of problem (2.1) if and only if (x, u, v, z, w) satisfies the following relation:*

$$g(x) = J_\rho^{M(\cdot,z))}(g(x) - \rho g(w) + \rho N(u, v)), \tag{3.1}$$

where $u \in \tilde{E}(x)$, $v \in \tilde{F}(x)$, $z \in \tilde{G}(x)$, $w \in \tilde{D}(x)$, and $\rho > 0$ is a constant.

PROOF. By the definition of the resolvent operator $J_\rho^{M(\cdot,z)}$ of $M(\cdot, z)$, we have that (3.1) holds if and only if $u \in \tilde{E}(x)$, $v \in \tilde{F}(x)$, $z \in \tilde{G}(x)$, and $w \in \tilde{D}(x)$ such that

$$g(x) - \rho g(w) + \rho N(u, v) \in g(x) + \rho M(g(x), z).$$

The above relations hold if and only if $u \in \tilde{E}(x)$, $v \in \tilde{F}(x)$, $z \in \tilde{G}(x)$, and $w \in \tilde{D}(x)$ such that

$$0 \in M(g(x), z) + g(w) - N(u, v).$$

Hence, (x, u, v, z, w) is a solution of problem (2.1) if and only if $u \in \tilde{E}(x)$, $v \in \tilde{F}(x)$, $z \in \tilde{G}(x)$, and $w \in \tilde{D}(x)$ are such that (3.1) holds.

REMARK 3.1. Theorem 3.1 is a variant form and generalization of Lemma 3.1 of Kazmi [4], Lemma 3.1 of Adly [2], Lemma 2.1 of Huang [3], and Theorem 3.1 of Ding [5,6]. This equation (3.1) can be written as

$$x = x - g(x) + J_\rho^{M(\cdot,z)}[g(x) - \rho g(w) + \rho N(u,v)]. \tag{3.2}$$

This fixed-point formulation enables us to suggest the following iterative algorithm.

ALGORITHM 3.1. Let $E, F, G, D : H \to \mathcal{F}(H)$ be closed fuzzy mappings satisfying Condition (A), and $\tilde{E}, \tilde{F}, \tilde{G}, \tilde{D} : H \to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings E, F, G, D, respectively. Let $N : H \times H \to H$ and $g : H \to H$ be single-valued mappings and let $M : H \times H \to 2^H$ be such that for each fixed $y \in H$, $M(\cdot, y) : H \to 2^H$ be a maximal monotone mapping satisfying $g(H) \cap \mathrm{dom}(M(\cdot, y)) \neq \emptyset$. For given $x_0 \in H$, $u_0 \in \tilde{E}(x_0)$, $v_0 \in \tilde{F}(x_0)$, $z_0 \in \tilde{G}(x_0)$, and $w_0 \in \tilde{D}(x_0)$, let $x_1 = x_0 - g(x_0) + J_\rho^{M(\cdot,z_0)}(g(x_0) - \rho g(w_0) + \rho N(u_0, v_0))$. By [17], there exist $u_1 \in \tilde{E}(x_1)$, $v_1 \in \tilde{F}(x_1)$, $z_1 \in \tilde{G}(x_1)$, and $w_1 \in \tilde{D}(x_1)$ such that

$$\begin{aligned}
\|u_0 - u_1\| &\leq (1+1)\tilde{H}\left(\tilde{E}(x_0), \tilde{E}(x_1)\right), \\
\|v_0 - v_1\| &\leq (1+1)\tilde{H}\left(\tilde{F}(x_0), \tilde{F}(x_1)\right), \\
\|z_0 - z_1\| &\leq (1+1)\tilde{H}\left(\tilde{G}(x_0), \tilde{G}(x_1)\right), \\
\|w_0 - w_1\| &\leq (1+1)\tilde{H}\left(\tilde{D}(x_0), \tilde{D}(x_1)\right).
\end{aligned}$$

Let $x_2 = x_1 - g(x_1) + J_\rho^{M(\cdot,z_1)}(g(x_1) - \rho g(w_1) + \rho N(u_1, v_1))$. By induction, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ satisfying

$$\begin{aligned}
&x_{n+1} = x_n - g(x_n) + J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) + \rho N(u_n, v_n)), \\
&u_n \in \tilde{E}(x_n), \quad \|u_n - u_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) \tilde{H}\left(\tilde{E}(x_n), \tilde{E}(x_{n+1})\right), \\
&v_n \in \tilde{F}(x_n), \quad \|v_n - v_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) \tilde{H}\left(\tilde{F}(x_n), \tilde{F}(x_{n+1})\right), \\
&z_n \in \tilde{G}(x_n), \quad \|z_n - z_{n+1}\| \leq \left(1 + (1+N)^{-1}\right) \tilde{H}\left(\tilde{G}(x_n), \tilde{G}(x_{n+1})\right), \\
&w_n \in \tilde{D}(x_n), \quad \|w_n - w_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) \tilde{H}\left(\tilde{D}(x_n), \tilde{D}(x_{n+1})\right), \qquad n = 0, 1, \dots,
\end{aligned} \tag{3.3}$$

where $\rho > 0$ is a constant.

THEOREM 3.2. *Let $E, F, G, D : H \to \mathcal{F}(H)$ be closed fuzzy mappings satisfying Condition (A) and $\tilde{E}, \tilde{F}, \tilde{G}, \tilde{D} : H \to \mathrm{CB}(H)$ be the set-valued mappings induced by the fuzzy mappings E, F, G, D, respectively. Let $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, and $\tilde{D}$ be ϵ-Lipschitz continuous, η-Lipschitz continuous, ζ-Lipschitz continuous, and λ-Lipschitz continuous, respectively. Let $g : H \to H$ be δ-strongly monotone and σ-Lipschitz continuous and let $N : H \times H \to H$ be α-relaxed Lipschitz with respect to $\tilde{E}$ and β-Lipschitz continuous in the first argument. Let $N(\cdot,\cdot)$ be ξ-Lipschitz continuous in the second argument. Suppose that for any $x, y, z \in H$,*

$$\left\| J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z) \right\| \leq \mu \|x - y\|, \tag{3.4}$$

and there exists a constant $\rho > 0$ such that

$$\begin{gathered}
k = 2\sqrt{1 - 2\delta + \sigma^2} + \mu\zeta, \qquad k + \rho\mu\zeta < 1, \qquad \xi\eta + \sigma\lambda < \alpha \leq \epsilon\beta, \\
\alpha > (1-k)(\xi\eta + \sigma\lambda) + \sqrt{(\epsilon^2\beta^2 - (\xi\eta + \sigma\lambda)^2)(2k - k^2)}, \\
\left| \rho - \frac{\alpha - (1-k)(\xi\eta + \sigma\lambda)}{\epsilon^2\beta^2 - (\xi\eta + \sigma\lambda)^2} \right| < \frac{\sqrt{[\alpha - (1-k)(\xi\eta + \sigma\lambda)]^2 - (\epsilon^2\beta^2 - (\xi\eta + \sigma\lambda)^2)(2k - k^2)}}{\epsilon^2\beta^2 - (\xi\eta + \sigma\lambda)^2}.
\end{gathered} \tag{3.5}$$

Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to x^*, u^*, v^*, z^*, and w^*, respectively, and $(x^*, u^*, v^*, z^*, w^*)$ is a solution of the nonlinear implicit quasivariational inclusion problem with fuzzy mappings (2.1).

PROOF. By Algorithm 3.1, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \Big\|x_n - g(x_n) + J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) + \rho N(u_n, v_n)) - x_{n-1} \\ &\quad - g(x_{n-1}). - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - \rho g(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\Big\| \\ &\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \Big\|J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) \\ &\quad + \rho N(u_n, v_n)) - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - \rho g(w_{n-1}) + \rho(u_{n-1}, v_{n-1}))\Big\|. \end{aligned} \tag{3.6}$$

Since g is δ-strongly monotone and σ-Lipschitz continuous, by using the technique of Noor [4], we have

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \le \sqrt{1 - 2\delta + \sigma^2}\, \|x_n - x_{n-1}\|. \tag{3.7}$$

By Lemma 2.1 and condition (3.4), we have

$$\begin{aligned} &\Big\|J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) + \rho N(u_n, v_n)) - J_\rho^{M(\cdot,z_{n-1})}(g(x_n) - \rho g(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\Big\| \\ &\quad \le \Big\|J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) + \rho N(u_n, v_n)) \\ &\qquad - J_\rho^{M(\cdot,z_n)}(g(x_{n-1}) - \rho g(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\Big\| \\ &\qquad + \Big\|J_\rho^{M(\cdot,z_n)}(g(x_{n-1}) - \rho g(w_{n-1}) + \rho N(u_{n-1}, v_{n-1})) \\ &\qquad - J_\rho^{M(\cdot,z_{n-1})}(g(x_{n-1}) - \rho g(w_{n-1}) + \rho N(u_{n-1}, v_{n-1}))\Big\| \\ &\quad \le \|g(x_n) - \rho g(w_n) + \rho N(u_n, v_n) - (g(x_{n-1}) - \rho g(w_{n-1}) \\ &\qquad - \rho N(u_{n-1}, v_{n-1}))\| + \mu\|z_n - z_{n-1}\| \\ &\quad \le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \|x_n - x_{n-1} + \rho(N(u_n, v_n) \\ &\qquad - N(u_{n-1}, v_n))\| + \rho\|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| \\ &\qquad + \rho\|g(w_n) - g(w_{n-1})\| + \mu\|z_n - z_{n-1}\|. \end{aligned} \tag{3.8}$$

Since $N(\cdot,\cdot)$ is α-relaxed Lipschitz with respect to $\tilde{E}$ and β-Lipschitz continuous in the first argument and $\tilde{E}$ is ϵ-Lipschitz continuous, we have

$$\begin{aligned} &\|x_n - x_{n-1} + \rho(N(u_n, v_n) - N(u_{n-1}, v_n))\|^2 \\ &\qquad = \|x_n - x_{n-1}\|^2 + 2\rho\langle N(u_n, v_n) \\ &\qquad\quad - N(u_{n-1}, v_n), x_n - x_{n-1}\rangle + \rho^2\|N(u_n, v_n) - N(u_{n-1}, v_n)\|^2 \\ &\qquad \le \|x_n - x_{n-1}\|^2 - 2\rho\alpha\|x_n - x_{n-1}\|^2 \\ &\qquad\quad + \rho^2\beta^2\left[\left(1 + n^{-1}\right)\tilde{H}\left(\tilde{E}(x_n), \tilde{E}(x_{n-1})\right)\right]^2 \\ &\qquad \le \left(1 - 2\rho\alpha + \rho^2\beta^2\epsilon^2\left(1 + n^{-1}\right)^2\right)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.9}$$

Using ξ-Lipschitz continuity of $N(\cdot,\cdot)$ in the second argument and η-Lipschitz continuity of $\tilde{F}$, we have

$$\begin{aligned} \|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| &\le \xi\|v_n - v_{n-1}\| \le \xi\left(1 + n^{-1}\right)\tilde{H}\left(\tilde{F}(x_n), \tilde{F}(x_{n-1})\right) \\ &\le \xi\eta\left(1 + n^{-1}\right)\|x_n - x_{n-1}\|. \end{aligned} \tag{3.10}$$

By the Lipschitz continuity of g and $\tilde{D}$, we have

$$\begin{aligned}\|g(w_n) - g(w_{n-1})\| &\leq \sigma\|w_n - w_{n-1}\| \leq \sigma\left(1+n^{-1}\right)\tilde{H}\left(\tilde{D}(x_n), \tilde{D}(x_{n-1})\right) \\ &\leq \sigma\left(1+n^{-1}\right)\lambda\|x_n - x_{n-1}\|.\end{aligned} \tag{3.11}$$

By the ζ-Lipschitz continuity of $\tilde{G}$, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq \left(1+n^{-1}\right)\tilde{H}\left(\tilde{G}(x_n), \tilde{G}(x_{n-1})\right) \\ &\leq \zeta\left(1+n^{-1}\right)\|x_n - x_{n-1}\|.\end{aligned} \tag{3.12}$$

By (3.6)–(3.12), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \Big[2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\alpha\rho+\epsilon^2\beta^2\rho^2(1+n^{-1})^2} \\ &\quad + \xi\eta\rho\left(1+n^{-1}\right) + \sigma\lambda\left(1+n^{-1}\right) + \mu\zeta\left(1+n^{-1}\right)\Big]\|x_n - x_{n-1}\| \\ &= (k_n + \lambda t_n(\rho))\|x_n - x_{n-1}\| \\ &= \theta_n\|x_n - x_{n-1}\|,\end{aligned} \tag{3.13}$$

where $k_n = 2\sqrt{1-2\delta+\sigma^2} + \mu\zeta(1+n^{-1})$, $t_n(\rho) = \sqrt{1-2\alpha\rho+\epsilon^2\beta^2\rho^2(1+n^{-1})^2} + (\xi\eta_\sigma\lambda)\rho(1+n^{-1})$, and $\theta_n = k_n + \lambda t_n(\rho)$. Letting $\theta = k + t(\rho)$, where $k = 2\sqrt{1-2\delta+\sigma^2} + \mu\zeta$ and $t(\rho) = \sqrt{1-2\alpha\rho+\epsilon^2\beta^2\rho^2} + (\xi\eta+\sigma\lambda)\rho$, we have that $k_n \to k$, $t_n(\rho) \to t(\rho)$, and $\theta_n \to \theta$ as $n \to \infty$. It follows from condition (3.5) that $\theta < 1$. Hence, $\theta_n < 1$ for n sufficiently large. Therefore, (3.13) implies that $\{x_n\}$ is a Cauchy sequence in H, and so we can assume that $x_n \to x^* \in H$ as $n \to \infty$. By the Lipschitz continuity of $\tilde{E}$, $\tilde{F}$, $\tilde{G}$, and $\tilde{D}$, we obtain

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \left(1+(1+n)^{-1}\right)\tilde{H}\left(\tilde{E}(x_{n+1}), \tilde{E}(x_n)\right) \leq \left(1+(1+n)^{-1}\right)\epsilon\|x_{n+1} - x_n\|, \\ \|v_{n+1} - v_n\| &\leq \left(1+(1+N)^{-1}\right)\tilde{H}\left(\tilde{F}(x_{n+1}), \tilde{F}(x_n)\right) \leq \left(1+(1+n)^{-1}\right)\eta\|x_{n+1} - x_n\|, \\ \|z_{n+1} - z_n\| &\leq \left(1+(1+n)^{-1}\right)\tilde{H}\left(\tilde{G}(x_{n+1}), \tilde{G}(x_n)\right) \leq \left(1+(1+n)^{-1}\right)\zeta\|x_{n+1} - x_n\|, \\ \|w_{n+1} - w_n\| &\leq \left(1+(1+n)^{-1}\right)\tilde{H}\left(\tilde{D}(x_{n+1}), \tilde{D}(x_n)\right) \leq \left(1+(1+n)^{-1}\right)\lambda\|x_{n+1} - x_n\|.\end{aligned}$$

It follows that $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ are also Cauchy sequences in H. We can assume that $u_n \to u^*$, $v_n \to v^*$, $z_n \to z^*$, and $w_n \to w^*$, respectively. Note that $u_n \in \tilde{E}(x_n)$, we have

$$\begin{aligned}d(u^*, \tilde{E}(x^*)) &\leq \|u^* - u_n\| + d\left(u_n, \tilde{E}(x_n)\right) + \tilde{H}\left(\tilde{E}(x_n), \tilde{E}(x^*)\right) \\ &\leq \|u^* - u_n\| + \epsilon\|x_n - x^*\| \to 0, \qquad \text{as } n \to \infty.\end{aligned}$$

Hence, we must have $u^* \in \tilde{E}(x^*)$. Similarly, we can show that $v^* \in \tilde{F}(x^*)$, $z^* \in \tilde{G}(x^*)$, and $w^* \in \tilde{D}(x^*)$. Hence, we have that $E_{x^*}(u^*) \geq a(x^*)$, $F_{x^*}(v^*) \geq b(x^*)$, $G_{x^*}(z^*) \geq c(x^*)$, and $D_{x^*}(w^*) \geq d(x^*)$. From $x_{n+1} = x_n - g(x_n) + J_\rho^{M(\cdot,z_n)}(g(x_n) - \rho g(w_n) + \rho N(u_n, v_n))]$, it follows that

$$g(x^*) = J_\rho^{M(\cdot,z^*)}(g(x^*) - \rho g(w^*) + \rho N(u^*, v^*)).$$

By Theorem 3.1, $(x^*, u^*, v^*, z^*, w^*)$ is a solution of problem (2.1).

REMARK 3.2. Theorem 3.2 is a general variant form of Theorem 3.1 of Huang [10]. It is also some improving variants of Theorem 3.3 of Ding [5] and Theorem 4.1 of Kazmi [4].

REMARK 3.3. Theorem 3.2 improves and generalizes Theorem 3.2 of Ding [15], Theorem 3.1 of Huang [3], Theorem 3.3 of Ding [5,6], Theorem 4.1 of Kazmi [4], Theorem 4.1 of Noor and Al-Said [12], and Theorem 3.1 of Verma [13,14] in several aspects.

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